Incidence Matrices of Polarized Projective Spaces*

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Abstract

In this paper, we first define a non-degenerate symmetric bilinear form on $\mathbb{F}_q^4$. Then we get an incidence matrix $\mathbf{G}$ of $\mathbb{F}_q^4$ by the bilinear form. By its corresponding quadratic form Q, the lines of $\mathbb{F}_q^4$ are classified as isotropic and anisotropic lines. Under this classification, we can get two sub-matrices of $\mathbf{G}$ and prove their 2-rank.

Key words and phrases: finite field, quadratic form, incidence matrix.

1 INTRODUCTION

Throughout this paper, q is an odd prime power, and $\mathbb{F}_q$ is the finite field with q elements. We endow the space $V = \mathbb{F}_q^4$ with a polarization, i.e., a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ over $\mathbb{F}_q$. Let Q be the corresponding quadratic form, without loss of generality, we always assume that $Q(X) = x_0^2 - x_1^2 + x_2^2 - \alpha x_3^2$, where α is a nonzero element in $\mathbb{F}_q$. As usual, we denote by $\mathbf{P}(V)$ the projective space of V, which is formed of the 1-dimensional subspaces of V. For each nonzero vector $X \in V$, let $\bar{X} \in \mathbf{P}(V)$ be the 1-dimensional subspace generated by X. The quadratic form Q gives rise to a matrix whose rows and columns are indexed by elements of $\mathbf{P}(V)$. This matrix is defined by the formula

$$\mathbf{G} : \mathbf{P}(V) \times \mathbf{P}(V) \to \{0, 1\}, \mathbf{G}(\bar{X}, \bar{Y}) = \begin{cases} 1, \bar{X} \perp \bar{Y}, \\ 0, \bar{X} \not\perp \bar{Y}. \end{cases}$$

We call that matrix the incidence matrix of $\mathbf{P}(V)$ associated to Q.

A vector $\bar{X} \in \mathbf{P}(V)$ is called *isotropic* or *anisotropic* according to whether $X \in V$ is *isotropic* or *anisotropic*. Let I (resp. A) be the set of isotropic (resp. anisotropic) vectors in $\mathbf{P}(V)$, and set

$$\mathbf{G}_{II} = \mathbf{G}|_{I\times I} \text{ and } \mathbf{G}_{AA} = \mathbf{G}|_{A\times A}.$$

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From [1,3], we have that the 2-rank of $\mathbf{G}_{II}$ is $q+1$ and the 2-rank of $\mathbf{G}_{AA}$ is q^2-1, respectively, in the case of $V=\mathbb{F}_q^3$. Moreover, based on computational evidence, it was conjectured in [1] that in the case of $V=\mathbb{F}_q^3$, $\mathbf{G}_{II}$ is always full rank and $\mathbf{G}_{AA}$ is full rank or its 2-rank is one less than its order according to n is odd or even. In this paper, we will prove the following theorem:

Theorem 1.1 *In the case of $V=\mathbb{F}_q^4$, $\mathbf{G}_{II}$ and $\mathbf{G}_{AA}$ are of full rank.*

2 The isotropic case

Proposition 2.1 *$\mathbf{G}_{II}$ is of full rank over $\mathbb{F}_2$.*

Lemma 2.2 *If $\bar{X},\bar{Y}$ are different isotropic lines such that $\langle X,Y\rangle=0$.Then any line on the plane that generated by $\bar{X}$ and $\bar{Y}$ is isotropic.*

Proof. The lines on that plane is in $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qY)$.Because $\bar{X}$,$\bar{Y}$ are isotropic lines and $\langle X,Y\rangle=0$. Consider with the bilinear form,we get $\langle k_1X+k_2Y,k_1X+k_2Y\rangle=0$, any $k_1,k_2\in\mathbb{F}_q$.Lemma has proved. ■

Lemma 2.3 *If there are three different isotropic lines on a plane,then any line on that plane is isotropic.*

Proof. Suppose $\bar{X},\bar{Y},\bar{Z}$ are different isotropic lines on a common plane.Then $\bar{Z}$ is an element in $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qY)$.From $\langle Z,Z\rangle=0$,we get $\langle X,Y\rangle=0$.From prop1.1,we can get the result. ■

From the bilinear form, consider the lines in the orthogonal complement of $\bar{X}$. We use $\bar{X}^{\perp}$ to represent the set of the lines in this hyperplane,it is $\bar{X}^{\perp}=\{\bar{W}|\langle W,X\rangle=0\}$.Then we get following lemma.

Lemma 2.4 *$\bar{X}$, $\bar{Y}$, $\bar{Z}$ are different lines,and $\bar{X}^{\perp}\cap\bar{Y}^{\perp}=\bar{X}^{\perp}\cap\bar{Y}^{\perp}\cap\bar{Z}^{\perp}$.Then $\bar{Z}$ is in $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qY)$.*

Proof. From the question we can get two groups of equations.Because they have same solutions, so the rank of coefficient matrix must be same. $\bar{X}$,$\bar{Y}$ are different lines, so the rank of the first coefficient matrix is 2, so is the second coefficient matrix.Then Z can be linear represented as k_1X+k_2Y, $k_1,k_2\in\mathbb{F}_q$, lemma has proved. ■

Lemma 2.5 *(1)let α be a nonzero square element of $\mathbb{F}_q$, that is $Q(X)=x_0^2-x_1^2+x_2^2-x_3^2$. We can get:any isotropic line $\bar{X}$, there are $2q+1$ isotropic lines and q^2-q anisotropic lines on $\bar{X}^{\perp}$;any anisotropic line $\bar{Y}$,there are $q+1$ isotropic lines and q^2 anisotropic lines on $\bar{Y}^{\perp}$.*
(2)let α be a non-square element of $\mathbb{F}_q$, that is $Q(X)=x_0^2-x_1^2+x_2^2-\alpha x_3^2$. We can get:any isotropic line $\bar{X}$,there is 1 isotropic line and q^2+q anisotropic lines on $\bar{X}^{\perp}$;any anisotropic line $\bar{Y}$,there are $q+1$ isotropic lines and q^2 anisotropic lines on $\bar{Y}^{\perp}$.

Proof. According to [1](lemma 1.9),we know if α is a nonzero square element,there are q^2+2q+1 isotropic lines and q^3-q anisotropic lines on $\mathbb{F}_q^4$.If α is a non-square element,there are q^2+1 isotropic lines and q^3+q anisotropic lines on $\mathbb{F}_q^4$.

(1)First we prove there are $2q+1$ isotropic lines on $\bar{X}^{\perp}$. $\langle X,X\rangle=0$,we want to find line $\bar{Z}$ such that $\langle X,Z\rangle=0$ and $\langle Z,Z\rangle=0$ set up at the same time.Let $X=(x_0,x_1,x_2,x_3)$,discuss below:

1.$x_0^2-x_1^2\neq 0$.

So,we get $x_0^2-x_1^2=x_3^2-x_2^2\neq 0$.Consider $P=(x_3-x_2,x_3-x_2,x_0-x_1,x_0-x_1)$, $Q=(x_3+x_2,x_3+x_2,-x_0+x_1,x_0-x_1)$ are the solutions of equations.We can verify that $P,Q\neq(0,0,0,0)$ and $\bar{X}$,$\bar{P}$,$\bar{Q}$ are different.Now,we get two different isotropic lines $\bar{P}$, $\bar{Q}$ on $\bar{X}^{\perp}$,and $\langle P,Q\rangle=-2(x_0^2-x_1^2)\neq 0$.We know $\bar{X}$, $\bar{P}$, $\bar{Q}$ are not in a common plane,otherwise,according to lemma 2.2,lines on the plane are all isotropic.In particular,line $\overline{P+Q}$ is an isotropic line.We can get $\langle P,Q\rangle=0$ by the bilinear form and $\langle P,P\rangle=\langle Q,Q\rangle=0$,contradiction.Because $\bar{X},\bar{P}$ are isotropic lines and $\langle X,P\rangle=0$,according to lemma 2.2,the lines on the plane that generated by $\bar{X}$ and $\bar{P}$ are all isotropic.It means the elements in $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qP)$ are isotropic.The same as plane generated by $\bar{X},\bar{Q}$,we get the elements in $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qQ)$ are isotropic.We can calculate each of $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qP)$ and $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qQ)$ has $q+1$ elements,and $\bar{X}$ is the unique element in both of them.So,we get $2q+1$ different isotropic lines in $\bar{X}^{\perp}$.Let's proof there isn't any other isotropic line on $\bar{X}^{\perp}$.Regard $\bar{X}$,$\bar{P}$,$\bar{Q}$ as basis of $\bar{X}^{\perp}$,the lines on $\bar{X}^{\perp}$ can be expressed as $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qP+\mathbb{F}_qQ)$.If there is another isotropic line on $\bar{X}^{\perp}$,let it be $\bar{R}$, $R=k_1X+k_2P+k_3Q$,according to $\langle R,R\rangle=0$,we can get $k_2=0$ or $k_3=0$,R is in $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qP)$ or $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qQ)$,contradiction.So,in this case,there are $2q+1$ isotropic lines on $\bar{X}^{\perp}$.

2.$x_0^2-x_1^2=0$,but $x_3^2-x_0^2\neq 0$.

Let $\dot{x_0}=x_3$, $\dot{x_1}=x_0$, $\dot{x_2}=x_1$, $\dot{x_3}=x_2$.According to $x_0^2-x_1^2+x_2^2-x_3^2=0$,we can get $\dot{x_0}^2-\dot{x_1}^2+\dot{x_2}^2-\dot{x_3}^2=0$,and $\dot{x_0}^2-\dot{x_1}^2\neq 0$.It's same as case 1.

3.$x_0^2-x_1^2=0$ and $x_3^2-x_0^2=0$.

It means $x_0^2=x_1^2=x_2^2=x_3^2$.Then $\bar{X}$ is one of the lines below:$\overline{(1,1,1,1)}$,$\overline{(1,1,1,-1)}$, $\overline{(1,1,-1,1)}$, $\overline{(1,1,-1,-1)}$,$\overline{(1,-1,1,1)}$, $\overline{(1,-1,1,-1)}$,$\overline{(1,-1,-1,1)}$,$\overline{(1,-1,-1,-1)}$. Similar to the proof above,any $\bar{X}$,if we find out two distinct isotropic line $\bar{P}$,$\bar{Q}$,and $\langle P,Q\rangle\neq 0$,then we can prove there are $2q+1$ isotropic lines on $\bar{X}^{\perp}$.

If $\bar{X}=\overline{(1,1,1,1)}$, let $\bar{P}=\overline{(1,1,-1,-1)}$, $\bar{Q}=\overline{(1,-1,-1,1)}$.
If $\bar{X}=\overline{(1,1,1,-1)}$, let $\bar{P}=\overline{(1,1,-1,1)}$, $\bar{Q}=\overline{(1,-1,-1,-1)}$.
If $\bar{X}=\overline{(1,1,-1,1)}$, let $\bar{P}=\overline{(1,1,1,-1)}$, $\bar{Q}=\overline{(1,-1,1,1)}$.
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If $\bar{X}=\overline{(1,-1,-1,-1)}$, let $\bar{P}=\overline{(1,-1,1,1)}$, $\bar{Q}=\overline{(1,1,1,-1)}$.

In conclusion,we proved when α is a nonzero square element, any isotropic line $\bar{X}$ on $\mathbb{F}_q^4$,there are $2q+1$ isotropic lines on $\bar{X}^{\perp}$.Because there are q^2+q+1 lines on $\bar{X}^{\perp}$, so

there are q^2-q anisotropic lines on $\bar{X}^{\perp}$.If $\bar{Y}$ is an anisotropic line , we can calculate that the number of isotropic lines on $\bar{Y}^{\perp}$ equals $\frac{(q+1)^2(q^2-q)}{q^3-q}$,it is $q+1$.The number of anisotropic lines on $\bar{Y}^{\perp}$ is q^2.(1) has proved.

(2)Now α is a non-square element and $Q(X)=x_0^2-x_1^2+x_2^2-\alpha x_3^2$.If $X=(x_0,x_1,x_2,x_3)$ and line $\bar{X}$ is an isotropic line,we want to prove $\bar{X}$ is the unique isotropic line on $\bar{X}^{\perp}$.Discuss below:

$1'$. $x_2^2\neq\alpha x_3^2$.It is said $x_0^2-x_1^2=x_2^2-\alpha x_3^2\neq 0$. $\bar{X}\in\bar{X}^{\perp}$,and we can find out $P=(x_1,x_0,\alpha x_3,x_2)$, $Q=(-x_1,-x_0,\alpha x_3,x_2)$. $\bar{P},\bar{Q}\in\bar{X}^{\perp}$.We can verify that $\bar{P},\bar{Q}$ are different lines and $\bar{X},\bar{P},\bar{Q}$ are not in a common plane.Notice that $\langle P,P\rangle=\langle Q,Q\rangle=(1-\alpha)(x_2^2-\alpha x_3^2)\neq 0$.So $\bar{P},\bar{Q}$ are anisotropic lines.Consider if there exsits an isotropic line $\bar{M}\in\mathbf{P}(\mathbb{F}_qP+\mathbb{F}_qQ)$.Let $M=k_1P+k_2Q$,obviously $k_1,k_2\neq 0$.From $\langle M,M\rangle=0$,we get $\langle k_1P+k_2Q,k_1P+k_2Q\rangle={k_1}^2\langle P,P\rangle+2k_1k_2\langle P,Q\rangle+k_2^2\langle Q,Q\rangle=0$.Regard this equation as a quadratic equation about k_1. $\Delta=16\alpha k_2^2(x_2^2-\alpha x_3^2)$ is a non-square element,so the equation has no solution.It means there is no isotropic line in $\mathbf{P}(\mathbb{F}_qP+\mathbb{F}_qQ)$.Any $\bar{M}\in\mathbf{P}(\mathbb{F}_qP+\mathbb{F}_qQ)$, $\langle M,M\rangle\neq 0$,but $\langle X,M\rangle=k_1\langle X,P\rangle+k_2\langle X,Q\rangle=0$.Consider $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qM)$,any $\overline{t_1X+t_2M}\in\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qM)$,if $t_2\neq 0$, $\langle t_1X+t_2M,t_1X+t_2M\rangle=t_1^2\langle X,X\rangle+2t_1t_2\langle M,X\rangle+t_2^2\langle M,M\rangle=t_2^2\langle M,M\rangle\neq 0$,so $\bar{X}$ is the unique isotropic line in $\mathbf{P}(\mathbb{F}_qX+\mathbb{F}_qM)$.And $\bar{M}$ is any line in $\mathbf{P}(\mathbb{F}_qP+\mathbb{F}_qQ)$,there are $q+1$ kinds of choices.So we can get $q+1$ planes,every of them has one isotropic line $\bar{X}$ and q anisotropic lines.Every two planes has only one common line,it is $\bar{X}$.So we get $q(q+1)$ different anisotropic lines on $\bar{X}^{\perp}$.Because $\bar{X}^{\perp}$ has q^2+q+1 lines,so they are the $q(q+1)$ different anisotropic lines and $\bar{X}$.Case $1'$ has proved.

$2'$. $x_2^2=\alpha x_3^2$.Because α is a non-square element,so $x_2=x_3=0$.Then we know $x_0=\pm x_1$.So there are only two lines satisfy this condition, $\overline{(1,1,0,0)}$ and $\overline{(1,-1,0,0)}$.Solve the equations we can get there is only one isotropic line $\overline{(1,1,0,0)}$ and q^2+q anisotropic lines on $\overline{(1,1,0,0)}^{\perp}$,there is only one isotropic line $\overline{(1,-1,0,0)}$ and q^2+q anisotropic lines on $\overline{(1,-1,0,0)}^{\perp}$.

In conclusion,we proved when α is a non-square element,any isotropic line $\bar{X}$ in $\mathbb{F}_q^4$,there is 1 isotropic line and q^2+q anisotropic lines on $\bar{X}^{\perp}$.If $\bar{Y}$ is an anisotropic line,we can calculate the number of isotropic lines on $\bar{Y}^{\perp}$ equals $\frac{(q^2+1)(q^2+q)}{q^3+q}$,it is equals $q+1$, and the number of anisotropic lines on $\bar{Y}^{\perp}$ is q^2.(2) has proved. ■

Lemma 2.6 *$\bar{P},\bar{Q}$ are different isotropic lines,then the number of isotropic lines on $\bar{P}^{\perp}\cap\bar{Q}^{\perp}$ is even.*

Proof. If there is no isotropic line on $\bar{P}^{\perp}\cap\bar{Q}^{\perp}$, the number must be 0,even. If $\bar{P}^{\perp}\cap\bar{Q}^{\perp}$ has some isotropic lines, then select $\bar{X}$,so $\langle X,X\rangle=0$,select $\bar{Y}\in\bar{P}^{\perp}\cap\bar{Q}^{\perp},\bar{Y}\neq\bar{X}$. $\bar{P}^{\perp}\cap\bar{Q}^{\perp}$ is a 2-dimensional subspace of V,so it can be generated by $\bar{X},\bar{Y}$.First we want to prove $\langle X,Y\rangle=0$ and $\langle Y,Y\rangle\neq 0$ can't set up at the same time.Otherwise, $\langle X,Y\rangle=0$ and $\langle Y,Y\rangle\neq 0$.So $\bar{X},\bar{Y}\in\bar{P}^{\perp}\cap\bar{Q}^{\perp}\cap\bar{X}^{\perp}$, $\bar{P}^{\perp}\cap\bar{Q}^{\perp}\subset\bar{P}^{\perp}\cap\bar{Q}^{\perp}\cap\bar{X}^{\perp}$.It means $\bar{P}^{\perp}\cap\bar{Q}^{\perp}=\bar{P}^{\perp}\cap\bar{Q}^{\perp}\cap\bar{X}^{\perp}$ According to lemma 2.4,we know $X=t_1P+t_2Q$,$t_1,t_2\in\mathbb{F}_q$.We can always obtain $\langle P,Q\rangle=0$ from $\langle X,X\rangle=0$.Now $\bar{P}^{\perp}\cap\bar{Q}^{\perp}=\mathbf{P}(\mathbb{F}_qP+\mathbb{F}_qQ)$.From lemma 2.2,lines in $\mathbf{P}(\mathbb{F}_qP+\mathbb{F}_qQ)$ are all isotropic lines.In particular,$\bar{Y}\in\bar{P}^{\perp}\cap\bar{Q}^{\perp}$,so

$\langle Y, Y\rangle = 0$,contradiction.It means $\langle X, Y\rangle = 0$ and $\langle Y, Y\rangle \neq 0$ can't set up at the same time.

Lines on $\bar{P}^\perp \cap \bar{Q}^\perp$ are in $\mathbf{P}(\mathbb{F}_q X + \mathbb{F}_q Y)$.Count the number of isotropic lines in $\mathbf{P}(\mathbb{F}_q X+\mathbb{F}_q Y)-\{\bar{X}\}$.We must solve equation $\langle t_1X+t_2Y, t_1X+t_2Y\rangle = 0$,$t_2 \neq 0$.Without loss of generality,let $t_2 = 1$.It is $2t_1\langle X, Y\rangle + \langle Y, Y\rangle = 0$.Discuss below:
case 1: $\langle X, Y\rangle = 0$,then $\langle Y, Y\rangle = 0$ must be set up.At this time,t_1 has q solutions.There are $q + 1$ isotropic lines on $\bar{P}^\perp \cap \bar{Q}^\perp$.Even.
case 2: $\langle X, Y\rangle \neq 0$.At this time,$t_1$ has only 1 solution.Then the number is 2,even. Lemma has proved. ■

Now we can prove prop 2.1.
1.α is a nonzero square element.We calculate $\mathbf{G}_{II}^2$.The element at position $(\bar{X}, \bar{X})$ of $\mathbf{G}_{II}^2$ equals the number of isotropic lines on $\bar{X}^\perp$.According to lemma 2.5(1),it is $2q + 1$,odd.The element at position $(\bar{X}, \bar{Y})$ $(\bar{X} \neq \bar{Y})$ of $\mathbf{G}_{II}^2$ is the number of isotropic lines on $\bar{X}^\perp \cap \bar{Y}^\perp$.According to lemma 2.6,it is even.So,$\mathbf{G}_{II}^2 \equiv I \pmod 2$.So,$\mathbf{G}_{II}$ is of full rank.
2.α is a non-square element.Accoding to lemma 2.5(2),the unique isotropic line on $\bar{X}^\perp$ is $\bar{X}$.So,$\mathbf{G}_{II} = I$.Full rank.The proposition has proved.

3 The anisotropic case

Proposition 3.1 *The 2-rank of $\mathbf{G}_{AA}$ is of full rank over $\mathbb{F}_2$.*

Lemma 3.2 *Suppose $\bar{P}_1$, $\bar{P}_2$ are different anisotropic lines.Then there is only one isotropic line on ${\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$ if and only if $\langle P_1, P_1\rangle\langle P_2, P_2\rangle = \langle P_1, P_2\rangle^2$.*

Proof.
$\Rightarrow$.If ${\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$ has only one isotropic line $\bar{Z}$,then,any other $\bar{X} \in {\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$, we have $\langle X, X\rangle \neq 0$. $\bar{Z}$ is the unique,so $\langle X + kZ, X + kZ\rangle \neq 0$, $k \in \mathbb{F}_q$.It means $\langle X, X\rangle + 2k\langle X, Z\rangle = 0$ has no solutions.So $\langle X, Z\rangle = 0$, $\bar{X} \in \bar{Z}^\perp$.And $\bar{Z} \in \bar{Z}^\perp$,so we get ${\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp \subset \bar{Z}^\perp$, it is ${\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp \cap \bar{Z}^\perp = {\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$.According to lemma 2.4, $\bar{Z}$ is on the plane that generated by $\bar{P}_1, \bar{P}_2$.Because $\bar{Z} \neq \bar{P}_1, \bar{P}_2$,so we can suppose $Z = t_1P_1 + t_2P_2$,$t_1, t_2 \in \mathbb{F}_q^*$.From $\bar{Z} \in {\bar{P}_1}^\perp$,we know $\langle P_1, Z\rangle = 0$,it is $t_1\langle P_1, P_1\rangle + t_2\langle P_1, P_2\rangle = 0$.From $\bar{Z} \in {\bar{P}_2}^\perp$,we know $\langle P_2, Z\rangle = 0$,it is $t_1\langle P_1, P_2\rangle + t_2\langle P_2, P_2\rangle = 0$. $t_1, t_2 \neq 0$,we can get $\langle P_1, P_1\rangle\langle P_2, P_2\rangle = \langle P_1, P_2\rangle^2$.
$\Leftarrow$.If $\langle P_1, P_1\rangle\langle P_2, P_2\rangle = \langle P_1, P_2\rangle^2$,so $\langle P_1, P_2\rangle \neq 0$.Let $\frac{\langle P_1,P_1\rangle}{\langle P_2,P_2\rangle} = \frac{\langle P_1,P_2\rangle}{\langle P_2,P_2\rangle} = t_0 \in \mathbb{F}_q^*$.We can obtain $\langle P_1, P_1 - t_0P_2\rangle = \langle P_2, P_1 - t_0P_2\rangle = 0$.So $\overline{P_1 - t_0P_2} \in {\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$.If there exsits another isotropic line $\bar{Y} \in {\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$,we can get $\langle Y, P_1\rangle = 0, \langle Y, P_2\rangle = 0$.It means $\langle Y, P_1 - t_0P_2\rangle = 0$.From lemma 2.1,we know the lines in ${\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$ are all isotropic. ${\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$ is a plane,the number of lines on it is $q+1$,and $\bar{P}_1, \bar{P}_2$ are anisotropic lines,from lemma 2.5,no matter α is a nonzero square element or a non-square element,the number of isotropic lines on ${\bar{P}_1}^\perp$ and ${\bar{P}_2}^\perp$ is $q + 1$.So ${\bar{P}_1}^\perp = {\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp = {\bar{P}_2}^\perp$.$\bar{P}_1 = \bar{P}_2$,contradiction.So $\overline{P_1 - t_0P_2}$ is the unique isotropic line on ${\bar{P}_1}^\perp \cap {\bar{P}_2}^\perp$.Lemma has proved. ■

From lemma 2.2,the number of isotropic lines on a plane is 0,1,2 or $q+1$.In lemma 3.2,we know if $\langle P_1, P_1\rangle\langle P_2, P_2\rangle \neq \langle P_1, P_2\rangle^2$,the number of isotropic lines on $\bar{P_1}^{\perp} \cap \bar{P_2}^{\perp}$ is even.

We know N denote the set of anisotropic line in $P(V)$.We also divide N into two parts, S and T. $S = \{\bar{X} \in P(V)|Q(X)$ *equals a nonzero square element in* $\mathbb{F}_q\}$, $T = \{\bar{X} \in P(V)|Q(X)$ *equals a nonsquare element in* $\mathbb{F}_q\}$.The lines in S are called square anisotropic lines and the lines in T are called non-square anisotropic lines.N is the disjoint union of S and T.Any $\bar{X} \in S$,we know $Q(X)$ equals a nonzero square element.We can find unique k_X such that $Q(k_X X) = 1$, $k_X \in \mathbb{F}_q^*$, $\bar{X}$ and $\overline{k_X X}$ are the same line.Similarly,if $\bar{Y} \in T$,we know $Q(X)$ equals a non-square element.We can find unique k_Y such that $Q(k_Y Y) = \beta$, $k_Y \in \mathbb{F}_q^*$,β is a fixed non-square element in $\mathbb{F}_q^*$, $\bar{Y}$ and $\overline{k_Y Y}$ is the same line.So in the following discussion,if $\bar{X}$ denotes a square anisotropic line,then we choose X such that $\langle X, X\rangle = 1$.Similarly,if $\bar{Y}$ denotes a non-square anisotropic line,then we choose Y such that $\langle Y, Y\rangle = \beta$.

Lemma 3.3 *$\bar{X}$ is a fixed square anisotropic line, such that $\langle X, X\rangle = 1$. Then the number of $\bar{Y}$ satisfy*

$$\begin{cases} \langle X, Y\rangle^2 = 1 \\ \langle Y, Y\rangle = 1 \end{cases} \tag{1}$$

is even ($\bar{Y} \neq \bar{X}$).

Proof. The equation has two parts:

$$\begin{cases} \langle X, Y\rangle = 1 \\ \langle Y, Y\rangle = 1 \end{cases} \tag{2}$$

and

$$\begin{cases} \langle X, Y\rangle = -1 \\ \langle Y, Y\rangle = 1 \end{cases} \tag{3}$$

If Y is a solution of (2),then $-Y$ is a solution of (3).Similarly,if Y is a solution of (3),then $-Y$ is a solution of (2).Consider $\bar{Y}$ and $\overline{-Y}$ are the same line,so we only solve equations (2). $\langle Y-X, X\rangle = \langle Y, X\rangle - \langle X, X\rangle = 0$, $\langle Y-X, Y-X\rangle = \langle Y, Y\rangle - 2\langle Y, X\rangle + \langle X, X\rangle = 0$.So $\overline{Y-X}$ is an isotropic line and $\overline{Y-X} \in \bar{X}^{\perp}$.From lemma 2.5,when $\bar{X}$ is an anisotropic line,there are $q+1$ isotropic lines on $\bar{X}^{\perp}$,no matter α is a nonzero square element or a non-square element.Let $\bar{X}_0, \bar{X}_1, \cdots, \bar{X}_q$ be the $q+1$ isotropic lines,then $\overline{Y-X} = \bar{X}_i$, $i \in \{0, 1, \cdots, q\}$.We get $Y = X + kX_i$, $k \in \mathbb{F}_q^*$,because $\bar{Y} \neq \bar{X}$.We can verify $Y = X + kX_i$ are the solutions and different $\bar{Y}$ represent different lines.In conclusion,the number of Y is $q^2 - 1$,even. ■

Lemma 3.4 *$\bar{X}, \bar{Y}$ are fixed square anisotropic line, $\bar{X} \neq \bar{Y}$. $\langle X, X\rangle = \langle Y, Y\rangle = 1$. Then*

the number of $\bar{Z}$ satisfy

$$\begin{cases} \langle X, Z\rangle^2 = 1 \\ \langle Y, Z\rangle^2 = 1 \\ \langle Z, Z\rangle = 1 \end{cases} \tag{4}$$

is even ($\bar{Z} \neq \bar{X}, \bar{Y}$).

Proof. The equation has four parts.

$$\begin{cases} \langle X, Z\rangle = 1 \\ \langle Y, Z\rangle = 1 \\ \langle Z, Z\rangle = 1 \end{cases} \tag{5}$$

and

$$\begin{cases} \langle X, Z\rangle = 1 \\ \langle Y, Z\rangle = -1 \\ \langle Z, Z\rangle = 1 \end{cases} \tag{6}$$

and

$$\begin{cases} \langle X, Z\rangle = -1 \\ \langle Y, Z\rangle = 1 \\ \langle Z, Z\rangle = 1 \end{cases} \tag{7}$$

and

$$\begin{cases} \langle X, Z\rangle = -1 \\ \langle Y, Z\rangle = -1 \\ \langle Z, Z\rangle = 1 \end{cases} \tag{8}$$

Consider that the solution of (5)and(8) represent the same lines and the solution of (6)and(7)represent the same lines,so we only solve (5)and(6).It is clear (5)(6)have no same solution.Discuss blow:

case1. $\langle X, Y\rangle \neq \pm 1$.First we find out nonzero constants $k_1 = (-1-\langle X, Y\rangle)/(1-\langle X, Y\rangle)$, $k_2 = 2/(1 - \langle X, Y\rangle)$,such that $k_1 + k_2 = 1, k_1 + \langle X, Y\rangle k_2 = -1$. If Z is a solution of (5),then $k_1 Z + k_2 X$ is a solution of (6),similarly,if Z' is a solution of (6),then $(Z' - k_2 X)/k_1$ is a solution of (5).In conclusion,if the solutions of (5) or (6) exsit ,the number of solution of (4) is even.if (5)and(6) both have no solutions,the number is zero,also even.Case 1 has proved.

case 2. $\langle X, Y\rangle = 1$.First we solve equation (5).We can get $\langle Z - X, X\rangle = 0$, $\langle Z - X, Y\rangle = 0$, $\langle Z - X, Z - X\rangle = 0$. $Z \neq X$,so $\overline{Z - X}$ is an isotropic line and $\overline{Z - X} \in \bar{X}^{\perp} \cap \bar{Y}^{\perp}$.Because $\bar{X}$ and $\bar{Y}$ are different anisotropic lines,according to lemma 3.2 , $\bar{X}^{\perp} \cap \bar{Y}^{\perp}$ has unique isotropic line,let it be $\bar{W}$.Then $\overline{Z - X} = \bar{W}$,we got $Z - X = tW, t \in \mathbb{F}_q^*$, $q - 1$ solutions.Notice that $Y - X$ satisfies $\overline{Y - X} \in \bar{X}^{\perp} \cap \bar{Y}^{\perp}$, $Y \neq X$,so Y has form $Y = X + t_0 W$,$t_0 \in \mathbb{F}_q^*$. Y must be one of the $q - 1$ solutions.Delete Y from the $q - 1$ solutions,we got $q - 2$ solutions of (5).

Second we solve equation (6).We can also get $\langle Z-X,X\rangle=0$, $\langle Z-X,Z-X\rangle=0$,$Z\neq X$,so $\overline{Z-X}$ is an isotropic line on $\bar{X}^{\perp}$.No matter α is a nonzero square element or a non-square element,there are $q+1$ isotropic lines in $\bar{X}^{\perp}$.Let them be $\bar{X}_0,\bar{X}_1,\cdots,\bar{X}_q$.So $\overline{Z-X}=\bar{X}_i$, $i\in\{0,1,\cdots,q\}$. Z has form $Z=X+kX_i$, $k\in\mathbb{F}_q^*$.Moreover,Z satisfies $\langle Z,Y\rangle=-1$,it means $\langle X+kX_i,Y\rangle=-1$, $k\langle X_i,Y\rangle=-2$.Consider $\bar{X}^{\perp}\cap\bar{Y}^{\perp}$ has unique isotropic line,let it be $\bar{X_0}$.It means $\langle X_0,Y\rangle=0$, $\langle X_i,Y\rangle\neq 0$, $i\in\{1,2,\cdots,q\}$.For X_i,$\langle X_i,Y\rangle\neq 0$,there exists unique $k=k_{X_i}$ such that $k_{X_i}\langle X_i,Y\rangle=-2$.We solve q solutions of (6).
In conclusion,in case 2,the number of solutions of (4) is $2q-2$,even.
case 3. $\langle X,Y\rangle=-1$.Let $X'=X,Y'=-Y,Z'=Z$.It's the same as case 2. Lemma has proved. ■

Lemma 3.5 *$\bar{X}$ is a fixed non-square anisotropic line,such that $\langle X,X\rangle=\beta$.The number of $\bar{Y}$ satisfy*

$$\begin{cases}\langle X,Y\rangle^2=\beta^2\\ \langle Y,Y\rangle=\beta\end{cases}\tag{9}$$

is even ($\bar{Y}\neq\bar{X}$).

The proof is similar to lemma 3.3.

Lemma 3.6 *$\bar{X}$,$\bar{Y}$ are fixed non-square anisotropic line, $\bar{X}\neq\bar{Y}$. $\langle X,X\rangle=\langle Y,Y\rangle=\beta$.The number of $\bar{Z}$ satisfy*

$$\begin{cases}\langle X,Z\rangle^2=\beta^2\\ \langle Y,Z\rangle^2=\beta^2\\ \langle Z,Z\rangle=\beta\end{cases}\tag{10}$$

is even ($\bar{Z}\neq\bar{X},\bar{Y}$).

The proof is similar to lemma 3.4.

Now we can prove prop 3.1 . In the above discussion,we know $N=S\cup T$.So we can give a partition of $\mathbf{G}_{AA}$.

$$\begin{pmatrix}\mathbf{G}_{SS} & \mathbf{G}_{ST}\\ \mathbf{G}_{TS} & \mathbf{G}_{TT}\end{pmatrix},\tag{11}$$

According to [1],we know the 4 submatrices are all square matrices.First we calculate $\mathbf{G}_{AA}^2$. The element at position $(\bar{X},\bar{X})$ of $\mathbf{G}_{AA}^2$ equals the number of anisotropic lines on $\bar{X}^{\perp}$.No matter α is a nonzero square element or a non-square element,it is q^2 , odd. The element at position $(\bar{X},\bar{Y})$ $(\bar{Y}\neq\bar{X})$ of $\mathbf{G}_{AA}^2$ equals the number of anisotropic lines on $\bar{X}^{\perp}\cap\bar{Y}^{\perp}$.In particular,from lemma 3.2,it is odd if and only if $\langle X,X\rangle\langle Y,Y\rangle=\langle X,Y\rangle^2$.When $\bar{X}\in S,\bar{Y}\in T$,we know $\langle X,X\rangle$ equals a square element and $\langle Y,Y\rangle$ equals a non-square element,so $\langle X,X\rangle\langle Y,Y\rangle=\langle X,Y\rangle^2$ can't set up. We obtain $\mathbf{G}_{AA}^2=$

$I + \begin{pmatrix} \mathbf{B}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_2 \end{pmatrix} \pmod 2$, The element on diagonal of $\mathbf{B}_1$ and $\mathbf{B}_2$ are all 0. Then $\mathbf{G}_{AA}^4 = I + \begin{pmatrix} \mathbf{B}_1^2 & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_2^2 \end{pmatrix} \pmod 2$.

We want to prove $\mathbf{B}_1^2 \equiv 0 \pmod 2, \mathbf{B}_2^2 \equiv 0 \pmod 2$. Calculate $\mathbf{B}_1^2$: The element at position $(\bar{X}, \bar{X})$:It equals the number of set $|\{\bar{Y} \neq \bar{X} | \langle X, X\rangle\langle Y, Y\rangle = \langle X, Y\rangle^2, \bar{Y} \in N\}|$.According to lemma 3.3,it is even.The element at position $(\bar{X}, \bar{Y})$ $(\bar{Y} \neq \bar{X})$:It equals the number of set $|\{\bar{Z} \neq \bar{X}, \bar{Y} | \langle Z, Z\rangle\langle X, X\rangle = \langle Z, X\rangle^2, \langle Z, Z\rangle\langle Y, Y\rangle = \langle Z, Y\rangle^2, \bar{Z} \in N\}|$.According to lemma 3.4,it is even.

Now we get $\mathbf{B}_1^2 \equiv 0 \pmod 2$.It's similar to $\mathbf{B}_2^2$,according to lemma 3.5 and lemma 3.6,we obtain $\mathbf{B}_2^2 \equiv 0 \pmod 2$.So,$\mathbf{G}_{AA}^4 \equiv I \pmod 2$,no matter α is a nonzero square element or a non-square element.It is clear the 2-rank of $\mathbf{G}_{AA}$ is of full rank.Prop 3.1 has proved.

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